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COMPUTATION OF THE PERTURBATIONS OF NEARLY CIRCULAR ORBITS, USING A NON-SINGULAR SET OF VECTORIAL ELEMENTS

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SUMMARY

A set of non-singular vectorial elements is used to obtain a new form of the variation of constants, which can be used in numerical integration of the perturbations of minor planets and satellites, even for orbits of very small eccentricity—without introducing the eccentricity as a "small divisor." The equations for treating the radiation pressure effect on nearly circular orbits of artificial satellites (of the Echo I type) are also derived.

CONTENTS

List of Symbols

M	The mean anomaly of the satellite
ω	The argument of perigee of the satellite
Ω	The longitude of the ascending node of the satellite
π	$= \omega + \Omega$
i	The inclination of the satellite orbit with respect to the equator
e	The eccentricity of the orbit of the satellite
a	The semi-major axis of the orbit of the satellite
p	$= a(1 - e^2)$
n	The mean motion of the satellite
f	The true anomaly of the satellite
E	The eccentric anomaly of the satellite
$\mathbf{r}$	The position vector of the satellite
$\mathbf{F}$	The disturbing force
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	The unit vectors of the moving system of coordinates
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	The unit vectors of the inertial system
$\mathbf{P}$	The unit vector directed from the central body toward the perigee
$\mathbf{Q}$	The unit vector standing normally to $\mathbf{P}$ and $\mathbf{k}$; $\mathbf{Q} = \mathbf{k} \times \mathbf{P}$
Ω	The disturbing function
χ	The position of the perigee in the orbital ideal system of coordinates
ψ	The difference between the osculating and the "mean" positions of perigee
λ	$= E + \psi$

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INTRODUCTION

In this paper a set of non-singular vectorial elements will be introduced which permits computation of the perturbations of orbits, even those with very small eccentricities, by means of numerical integration. The classical method of treating the variation of elliptic elements, as well as some methods developed more recently (References 1, 2, and 3), suffers from the inclusion of the eccentricity in the denominator; this makes computations difficult for the case of nearly circular orbits.

The launching of satellites into orbits with small eccentricities has brought this problem to attention again. The solving of this question requires the development of new forms of the variation of arbitrary constants, applicable to cases of both moderate and small eccentricities. Furthermore, these equations for the variation of constants must possess a symmetry in order to facilitate programming and make the use of machines more efficient.

In a previous work (Reference 2) the author made the following statement about the eccentricity:

> "It is impossible to remove this factor from the denominator of . . . (the equation for $d\Delta M/dt$) . . . or some corresponding equation, which controls the angular position of the planet in its orbit unless we resort to series expansions such as those used by Delaunay."

The author now revokes this statement, and the present discussion will show that this small divisor in the computation of special perturbations is in fact removable.

Recently S. Herrick (Reference 4) suggested a method that is valid for small eccentricities. The new method which will be set forth here makes use of Gibbs' rotation vector (Reference 5) and the vector of the Hamiltonian integral.

The difficulty connected with orbits of small eccentricity comes from two sources. The first source is a tendency always to deduce a unit vector directed toward the osculating perigee. The determination of this unit vector, unfortunately, is always connected with a division by the eccentricity. This difficulty can be avoided by representing the position vector of the body in terms of the Gibbs' and Hamiltonian vectors. The second source of difficulty is in the use of the disturbed mean anomaly. This difficulty can be removed by combining the perturbations in the perigee position with the eccentric anomaly, and not with the mean disturbed anomaly, and rewriting Kepler's equation in terms of the disturbed mean longitude.

The difficulty in computing the radiation pressure effect for a circular orbit initiated the development of this method. However, this method is general and can also be used to compute the perturbations of planets or satellites caused by gravitational forces.

PERTURBATIONS OF MINOR PLANETS

Let $\mathbf{k}$ be the unit vector normal to the osculating orbit plane, $\mathbf{P}$ the unit vector directed from the central body to the osculating perihelion, and $\mathbf{Q} = \mathbf{k} \times \mathbf{P}$. The values of these vectors for the initial instant will be designated by $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$. The equation of the motion of a minor planet is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3} + \mathbf{F}, \tag{1}$$

where $\mathbf{r}$ is the position vector of the satellite, t is time, and $\mathbf{F}$ is the disturbing force. The osculating orbit plane can be considered a rigid body with its instantaneous axis of rotation coinciding with the radius vector. Imagine a system of coordinates rigidly connected with the osculating orbital plane and rotating with it. Hansen (Reference 6) labeled such a system "ideal." It has the property that the equation of motion of the planet, relative to this system, retains the form of Equation 1. We arbitrarily choose two axes of this system to be in the orbit plane and the third axis to be normal to it. Let χ be the longitude of the osculating perihelion with respect to the ideal system of coordinates.

The initial position of the perigee, originally defined by the vector $\mathbf{i}_0$, in the inertial system, can be imagined to participate in the rotation of the orbit plane. At time t, the vector $\mathbf{i}$ is the unit vector to the initial position of perigee. Its position is invariable with respect to the ideal system, according to the definition of $\mathbf{i}$, and will be designated π_0.

Let $\mathbf{j} = \mathbf{k} \times \mathbf{i}$. The system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is deduced by a plain rotation of the system $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ without sliding. Vector $\mathbf{P}$, however, in addition to rotating, slides in the orbit plane.

The position vector of the planet is

$$\mathbf{r} = \mathbf{P}\, a(\cos E - e) + \mathbf{Q}\, a\sqrt{1 - e^2}\, \sin E, \tag{2}$$

where a is the semi-major axis, e is the eccentricity, E is the eccentric anomaly of the satellite, and

$$\mathbf{P} = + \mathbf{i} \cos(\chi - \pi_0) + \mathbf{j} \sin(\chi - \pi_0), \tag{3}$$

$$\mathbf{Q} = - \mathbf{i} \sin(\chi - \pi_0) + \mathbf{j} \cos(\chi - \pi_0). \tag{4}$$

Introducing the Hamiltonian vector

$$\mathbf{g} = \frac{h}{h_0} e \mathbf{Q}$$

$$= - \mathbf{i} v + \mathbf{j} u \tag{5}$$

where

$$h = \frac{1}{\sqrt{p}}, \quad h_0 = \frac{1}{\sqrt{p_0}}, \quad u = \frac{h}{h_0} e \cos(\chi - \pi_0), \quad \text{and} \quad v = \frac{h}{h_0} e \sin(\chi - \pi_0), \tag{6}$$

and putting

$$\lambda = E + (\chi - \pi_0), \tag{7}$$

$$\boldsymbol{\rho} = a(\mathbf{i} \cos \lambda + \mathbf{j} \sin \lambda), \tag{8}$$

we deduce (from Equations 2-8):

$$\mathbf{r} = \boldsymbol{\rho} - \frac{h_0^2}{h^2} \frac{\mathbf{g}\, \mathbf{g} \cdot \boldsymbol{\rho}}{1 + \sqrt{1 - e^2}} + \left(a \frac{h_0}{h} \right) \mathbf{k} \times \mathbf{g}. \tag{9}$$

Equation 9 shows that no computation of the unit vector $\mathbf{P}$ is necessary since the perturbations in the perigee position are combined with the eccentric anomaly, and the position vector of the body is expressible in terms of the Hamiltonian vector directly. Consequently no division by e takes place.

For a minor planet we make use of Gibbs' rotation vector to represent the rotation of the orbit plane from its initial position $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ to its position $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ at time t. The use of this vector instead of the elements Ω and i removes the difficulty connected with a small inclination. In addition, the matrix of rotation is obtained in a more direct way, without resorting to trigonometric transformations. To some degree this idea is an extension of the idea used by B. Strömgren (Reference 1) in his method for special perturbations. Strömgren used the integrated value of the angular velocity of rotation of the orbit plane as a substitute for Gibbs' vector. However, this approximation does not accurately represent the rotation of the orbit plane from its initial position, and as a result only the perturbations of the first order can be easily obtained with Strömgren's method. Let θ, ϕ, and ψ be, respectively, the inclination of the osculating orbit, the longitude of the ascending node, and the argument of perigee with respect to the reference system $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$.

Set

$$s_1 = \frac{\cos\frac{\phi - \psi}{2}}{\cos\frac{\phi + \psi}{2}} \tan\frac{\theta}{2}, \tag{10}$$

$$s_2 = \frac{\sin\frac{\phi - \psi}{2}}{\cos\frac{\phi + \psi}{2}} \tan\frac{\theta}{2}, \tag{11}$$

$$s_3 = \tan\frac{\phi + \psi}{2}, \tag{12}$$

$$\mathbf{s} = s_1\mathbf{i}_0 + s_2\mathbf{j}_0 + s_3\mathbf{k}_0, \tag{13}$$

$$s^2 = s_1^2 + s_2^2 + s_3^2.$$

The ratios

$$\xi_1 = \sin\frac{\theta}{2}\cos\frac{\phi - \psi}{2} = \frac{s_1}{\sqrt{1 + s^2}},$$

$$\xi_2 = \sin\frac{\theta}{2}\sin\frac{\phi - \psi}{2} = \frac{s_2}{\sqrt{1 + s^2}},$$

$$\xi_3 = \cos\frac{\theta}{2}\sin\frac{\phi + \psi}{2} = \frac{s_3}{\sqrt{1 + s^2}},$$

$$\xi_4 = \cos\frac{\theta}{2}\cos\frac{\phi + \psi}{2} = \frac{1}{\sqrt{1 + s^2}},$$

are Euler's parameters of rotation. In terms of $\mathbf{s}$ the matrix of rotation Γ, representing the rotation of $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ to position $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, can be written in the form (Reference 5)

$$\Gamma = I + \frac{2}{1 + s^2}\left[\mathbf{s}\times I + \mathbf{s}\times(\mathbf{s}\times I)\right] = \mathbf{i}\,\mathbf{i}_0 + \mathbf{j}\,\mathbf{j}_0 + \mathbf{k}\,\mathbf{k}_0, \tag{14}$$

or in the form

$$\Gamma = \frac{1 - s^2}{1 + s^2} I + \frac{2\mathbf{s}\times I + 2\mathbf{s}\,\mathbf{s}}{1 + s^2}, \tag{15}$$

where I is the idemfactor. Equation 14 or 15 should be used instead of the development of Γ into an infinite series (Reference 7). Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two Gibbs' vectors. If the rotation defined by $\mathbf{S}_2$ geometrically follows the rotation defined by $\mathbf{S}_1$, then the Gibbs' vector of the combined rotation is given by the formula

$$\mathbf{S}_3 = \frac{\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_2\times\mathbf{S}_1}{1 - \mathbf{S}_1\cdot\mathbf{S}_2}. \tag{16}$$

Let $\boldsymbol{\Omega}$ be the instantaneous angular velocity of rotation of the osculating orbit plane and ω_1, ω_2, and 0 its projections in the directions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. We have

$$\begin{aligned}\boldsymbol{\Omega} &= \mathbf{i}\omega_1 + \mathbf{j}\omega_2 \\ &= \frac{h}{h_0}(h_0\,\mathbf{F}\cdot\mathbf{k})\mathbf{r}.\end{aligned} \tag{17}$$

Put

$$\begin{aligned}\boldsymbol{\omega} &= \mathbf{i}_0\omega_1 + \mathbf{j}_0\omega_2 \\ &= \Gamma^{-1}\cdot\boldsymbol{\Omega}.\end{aligned} \tag{18}$$

Replacing $\mathbf{s}$ by $-\mathbf{s}$ in Equation 14 we have:

$$\Gamma^{-1} = \mathbf{i}_0\,\mathbf{i} + \mathbf{j}_0\,\mathbf{j} + \mathbf{k}_0\,\mathbf{k} = I - \frac{2}{1 + s^2}\left[\mathbf{s}\times I - \mathbf{s}\times(\mathbf{s}\times I)\right], \tag{19}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} - \frac{2}{1 + s^2}\left[\mathbf{s}\times\boldsymbol{\Omega} - \mathbf{s}\times(\mathbf{s}\times\boldsymbol{\Omega})\right], \tag{20}$$

$$\mathbf{\Omega} = \boldsymbol{\omega} + \frac{2}{1 + s^2}\left[\mathbf{s}\times\boldsymbol{\omega} + \mathbf{s}\times(\mathbf{s}\times\boldsymbol{\omega})\right]$$

$$= \Gamma\cdot\boldsymbol{\omega}\,. \tag{21}$$

Let $\mathbf{s}$ and $\mathbf{s}+d\mathbf{s}$ be the Gibbs' vectors defining the rotation of the osculating orbit plane from the moment t_0 to the moments t and $t+dt$, respectively. The infinitesimal matrix of rotation which brings the osculating plane from its position at time t to its position at time $t+dt$ is $I + \boldsymbol{\omega}\times I\,dt$. Since for an infinitesimal rotation, higher order terms can be neglected, then $\Gamma = I + 2\mathbf{s}\times I$, and from this we deduce that the corresponding Gibbs' vector is $(1/2)\boldsymbol{\omega}\,dt$. Substituting $S_1 = (1/2)\boldsymbol{\omega}\,dt$, $S_2 = \mathbf{s}$, and $S_3 = \mathbf{s} + d\mathbf{s}$ into Equation 16, we deduce that

$$\frac{d\mathbf{s}}{dt} = \frac{1}{2}\boldsymbol{\omega} + \frac{1}{2}\mathbf{s}\times\boldsymbol{\omega} + \frac{1}{2}\mathbf{s}\,\mathbf{s}\cdot\boldsymbol{\omega}$$

or

$$\frac{d\mathbf{s}}{dt} = \frac{1}{2}(I + \mathbf{s}\times I + \mathbf{s}\,\mathbf{s})\cdot\boldsymbol{\omega}\,. \tag{22}$$

We find from Equation 14, $(1/2)(1 + s^2)(\Gamma + I) = I + \mathbf{s}\times I + \mathbf{s}\,\mathbf{s}$, and from Equations 21, 22, and the preceding equation,

$$\frac{d\mathbf{s}}{dt} = \frac{1 + s^2}{4}(\mathbf{\Omega} + \boldsymbol{\omega}). \tag{23}$$

Multiplying Equation 22 by

$$\frac{2}{1 + s^2}(I - \mathbf{s}\times I)$$

and considering

$$I\cdot\mathbf{A} = \mathbf{A},$$

$$\mathbf{A}\times I\cdot\mathbf{B} = \mathbf{A}\times\mathbf{B},$$

$$(\mathbf{s}\times I)\cdot(\mathbf{s}\times I) = \mathbf{s}\,\mathbf{s} - s^2 I,$$

we deduce

$$\boldsymbol{\omega} = \frac{2}{1 + s^2}(I - \mathbf{s}\times I)\frac{d\mathbf{s}}{dt}\,,$$

or

$$\boldsymbol{\omega} = \frac{2}{1 + s^2}\left(\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{s}}{dt}\times\mathbf{s}\right).$$

This vectorial formula is identical with the classic scalar formulas:

$$\frac{\omega_x}{2} = + \xi_4 \frac{d\xi_1}{dt} + \xi_3 \frac{d\xi_2}{dt} - \xi_2 \frac{d\xi_3}{dt} - \xi_1 \frac{d\xi_4}{dt} ,$$

$$\frac{\omega_y}{2} = - \xi_3 \frac{d\xi_1}{dt} + \xi_4 \frac{d\xi_2}{dt} + \xi_1 \frac{d\xi_3}{dt} - \xi_2 \frac{d\xi_4}{dt} ,$$

$$\frac{\omega_z}{2} = + \xi_2 \frac{d\xi_1}{dt} - \xi_1 \frac{d\xi_2}{dt} + \xi_4 \frac{d\xi_3}{dt} - \xi_3 \frac{d\xi_4}{dt} .$$

From the formula

$$\boldsymbol{\Omega} = \frac{h}{h_0} (h_0\mathbf{F}\cdot\mathbf{k})\mathbf{r}$$

and Equation 20 we obtain:

$$\begin{aligned} \boldsymbol{\omega} &= \frac{h}{h_0} (h_0\mathbf{F}\cdot\mathbf{k})(\Gamma^{-1}\cdot\mathbf{r}) \\ &= \frac{h}{h_0} (h_0\mathbf{F}\cdot\mathbf{k}) \left[\mathbf{r} - \frac{2\mathbf{s}\times\mathbf{r} - 2\mathbf{s}\times(\mathbf{s}\times\mathbf{r})}{1 + s^2}\right] \\ &= \left(\frac{h}{h_0}\right)(h_0\mathbf{F}\cdot\mathbf{k}) \frac{\mathbf{r}\,(1 - s^2) - 2\mathbf{s}\times\mathbf{r} + 2\mathbf{s}\,\mathbf{s}\cdot\mathbf{r}}{1 + s^2} . \end{aligned} \tag{24}$$

Substituting Equations 17 and 24 into Equation 23 we have, as a final result,

$$\frac{d\mathbf{s}}{dt} = \frac{1}{2}\frac{h}{h_0} (h_0\mathbf{F}\cdot\mathbf{k})(\mathbf{r} + \mathbf{r}\times\mathbf{s} + \mathbf{s}\,\mathbf{s}\cdot\mathbf{r}). \tag{25}$$

For Strömgren's method

$$\frac{d\mathbf{s}}{dt} = \frac{1}{2}\frac{h}{h_0} (h_0\mathbf{F}\cdot\mathbf{k})\mathbf{r},$$

which evidently represents only a first approximation. The fact that two terms of

Equation 25 are missing from the equation for Strömgren's method makes the computation of accurate perturbations by Stromgren's method impossible.

The equation for the variation of $\mathbf{g}$ has been deduced previously (Reference 2). In the present case it takes the form:

$$\frac{d\mathbf{g}}{dt} = \left(1 + \frac{r}{p}\right)\frac{\mathbf{F}}{h_0} - \frac{\mathbf{r}\ \mathbf{r}\cdot\mathbf{F}}{rph_0} - \frac{2r}{ph_0}\ \mathbf{k}\ \mathbf{k}\cdot\mathbf{F}. \tag{26}$$

The computation of $\mathbf{s}$ and $\mathbf{g}$ by means of numerical integration is performed in the inertial system directly. By setting

$$\boldsymbol{\rho}_0^* = \mathbf{i}_0 \cos\lambda + \mathbf{j}_0 \sin\lambda,$$

we have for $\boldsymbol{\rho}^*$, the unit vector in the direction of $\boldsymbol{\rho}$ in the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ system,

$$\boldsymbol{\rho}^* = \boldsymbol{\rho}_0^* + \frac{2}{1 + s^2}\left[\mathbf{s}\times\boldsymbol{\rho}_0^* + \mathbf{s}\times(\mathbf{s}\times\boldsymbol{\rho}_0^*)\right],$$

and for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,

$$\mathbf{U} = \mathbf{U}_0 + \frac{2}{1 + s^2}\left[\mathbf{s}\times\mathbf{U}_0 + \mathbf{s}\times(\mathbf{s}\times\mathbf{U}_0)\right],$$

where $\mathbf{U} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{U}_0 = \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$. Finally, in the inertial system,

$$\mathbf{r} = a\boldsymbol{\rho}^* - a\,\frac{{h_0}^2}{h^2}\left(\frac{\mathbf{g}\ \mathbf{g}\cdot\boldsymbol{\rho}^*}{1 + \sqrt{1 - e^2}}\right) + a\,\frac{h_0}{h}\,\mathbf{k}\times\mathbf{g}. \tag{27}$$

In addition we have $u = +\mathbf{g}\cdot\mathbf{j}$, $v = -\mathbf{g}\cdot\mathbf{i}$, and also the classical result:

$$\frac{d}{dt}\left(\frac{h_0}{h}\right) = h_0\,\mathbf{F}\cdot\mathbf{k}\times\mathbf{r} \tag{28}$$

and

$$n = \left(\frac{h^2}{{h_0}^2} - g^2\right)^{3/2} {h_0}^3, \tag{29}$$

where n is the mean motion of the satellite.

If preferred, an equation for n can be formed and integrated directly. The classical formula:

$$\frac{d\Delta L_1}{dt} = -\frac{2}{\sqrt{a}}\mathbf{r}\cdot\mathbf{F} + \left(1 - \sqrt{1 - e^2}\right)\frac{d\pi}{dt} + 2\sqrt{1 - e^2}\,\sin^2\frac{1}{2}i\,\frac{d\Omega}{dt},$$

for the perturbations in the mean longitude takes the form:

$$\frac{d\Delta L_1}{dt} = -\frac{2}{\sqrt{a}}\mathbf{r}\cdot\mathbf{F} + \left(1 - \sqrt{1 - e^2}\right)\frac{d\chi}{dt} \tag{30}$$

in the ideal system of coordinates. From Equations 6-9 it follows that

$$\frac{d\chi}{dt} = \frac{u\,\frac{dv}{dt} - v\,\frac{du}{dt}}{u^2 + v^2}. \tag{31}$$

From

$$\frac{d\mathbf{g}}{dt} = -\mathbf{i}\frac{dv}{dt} + \mathbf{j}\frac{du}{dt} + \boldsymbol{\Omega}\times\mathbf{g}$$

there follows, taking Equation 31 into account,

$$\frac{d\chi}{dt} = \frac{1}{g^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt};$$

and finally, from Equation 30,

$$\frac{d\Delta L_1}{dt} = -\frac{2}{\sqrt{a}}\mathbf{r}\cdot\mathbf{F} + \frac{1}{1 + \sqrt{1 - e^2}}\,\frac{h_0^2}{h^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt}. \tag{32}$$

Kepler's equation takes the form:

$$\lambda - \frac{h_0}{h}(u \sin\lambda - v\cos\lambda) = M_0 + n_0(t - t_0) + \iint\frac{dn}{dt}\,dt^2 + \Delta L_1, \tag{33}$$

or the form:

$$\lambda - \frac{h_0}{h}\,\mathbf{g}\cdot\boldsymbol{\rho}^* = M_0 + n_0(t - t_0) + \iint\frac{dn}{dt}\,dt^2 + \Delta L_1.$$

As we see, the use of vectors $\mathbf{s}$ and $\mathbf{g}$ instead of the elliptical elements or instead of the elements $\mathbf{P}$ and $\mathbf{k}$ permits the computation of perturbations even for nearly circular orbits with no small numerical divisor.

The differential equations for $\mathbf{s}$ and $\mathbf{g}$ admit an integral

$$\mathbf{g} \cdot \mathbf{k} = 0$$

which can be used to check the computation.

PERTURBATIONS OF SATELLITES

In the case of either an artificial satellite or a natural planetary satellite, a method of numerical integration of perturbations will take a most convenient form if it is based on the mutual separation of secular, long period, and short period effects. A rotating system of coordinates must be chosen as the basic reference frame in such a way that its motion contains the secular and long period motions of the node and of the perigee, at least to a reasonably good approximation. Such a choice is necessary to prevent a fast accumulation of perturbations as well as to avoid the necessity of a rectification of elements after the perturbations become large.

Numerical integration permits the drag effect to be included from the very start, as well as the cross action between the oblateness and the drag effect. The case of critical orbit inclination can also be treated by means of numerical integration, and a convenient scheme for doing this has been reported by D. Fisher (Reference 8). The inclusion of tesseral harmonics becomes less complicated in this method than in a general theory.

The mean motions of the node and perigee and the long period effects can be obtained to a reasonable approximation from existing analytical or semi-analytical theory. The scheme given here is designed to tie the author's semi-analytical theory to the process of numerical integration. However, other theories, with small modifications, can serve the same purpose. In particular, the polynomial representation of secular and long period effects published by the Smithsonian Astrophysical Observatory can be of great assistance.

In addition to the rotating system mentioned above, a certain ideal system of coordinates is to be associated with the moving orbit plane. Let us designate by $\int n_0 \, y \, dt$ the perturbations of the position of perigee, which is included from the very start, and let π_0 be the original position of perigee with respect to the ideal system. The "mean" perigee is determined by the expression $\pi_0 + \int n_0 y \, dt$. The osculating perigee describes oscillations about its mean position. Let $\mathbf{i}$ be the unit vector directed toward the mean position of perigee, and $\mathbf{j} = \mathbf{k} \times \mathbf{i}$ as before. The position vector of the satellite is given by

$$\mathbf{r} = \mathbf{i} r \cos(f + \chi - \pi_0 - \int n_0 y \, dt) + \mathbf{j} r \sin(f + \chi - \pi_0 - \int n_0 y \, dt). \tag{34}$$

The Hamiltonian vector is introduced by the equation

$$\mathbf{g} = -\mathbf{i}v + \mathbf{j}u \tag{35}$$

where, in this case,

$$u = \frac{h}{h_0}e\cos(\chi - \pi_0 - \int n_0 y\,dt), \tag{36}$$

$$v = \frac{h}{h_0}e\sin(\chi - \pi_0 - \int n_0 y\,dt). \tag{37}$$

These values of u and v were introduced by Brown (Reference 9) into Hansen's theory to deduce the value of the basic function W. With these new definitions of $\mathbf{i}$, $\mathbf{j}$, u, v, and $\mathbf{g}$, we have formulas of the same type as before:

$$\mathbf{r} = \boldsymbol{\rho} - \frac{h_0^2}{h^2}\,\frac{\mathbf{g}\;\mathbf{g}\cdot\boldsymbol{\rho}}{1 + \sqrt{1 - e^2}} + a\frac{h_0}{h}\,\mathbf{k}\times\mathbf{g}, \tag{38}$$

$$\boldsymbol{\rho} = \mathbf{i}a\cos\lambda + \mathbf{j}a\sin\lambda, \tag{39}$$

where λ is now defined by the equation

$$\lambda = E + \chi - \pi_0 - \int n_0 y\,dt. \tag{40}$$

Designating the radial component of the disturbing force by S, the normal component to the radius vector by T, and the orthogonal component by Z, we have (Reference 9):

$$\frac{du}{dt} = +\frac{S}{h_0}\sin(f + \chi - \pi_0 - \int n_0 y\,dt) + \left(1 + \frac{r}{p}\right)\frac{T}{h_0}\cos(f + \chi - \pi_0 - \int n_0 y\,dt) + n_0 y v, \tag{41}$$

$$\frac{dv}{dt} = -\frac{S}{h_0}\cos(f + \chi - \pi_0 - \int n_0 y\,dt) + \left(1 + \frac{r}{p}\right)\frac{T}{h_0}\sin(f + \chi - \pi_0 - \int n_0 y\,dt) - n_0 y u, \tag{42}$$

where f is the true anomaly of the satellite. For the derivative of $\mathbf{g}$ with respect to the moving axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have from Equations 41 and 42:

$$\frac{d\mathbf{g}}{dt} = +\frac{S}{h_0}\,\mathbf{r}^* + \frac{T}{h_0}\left(1 + \frac{r}{p}\right)\mathbf{k}\times\mathbf{r}^* - n_0 y\,\mathbf{k}\times\mathbf{g}, \tag{43}$$

where $\mathbf{r}^*$ is the unit vector in the direction of $\mathbf{r}$. In order to obtain the derivative of $\mathbf{g}$ in the inertial system, the effect of rotation of the orbit plane about the instantaneous radius vector and the effect of rotation of $\mathbf{i}$ and $\mathbf{j}$ about $\mathbf{k}$ must be added to Equation 43. The combined effect of these two rotations is $(n_0 y\mathbf{k} + hZ\mathbf{r})\times\mathbf{g}$ and we deduce:

$$\frac{d\mathbf{g}}{dt} = \frac{S\mathbf{r}^*}{h_0} + \frac{T}{h_0}\left(1 + \frac{r}{p}\right)\mathbf{k}\times\mathbf{r}^* + \frac{h}{h_0} Zh_0\mathbf{r}\times\mathbf{g}. \tag{44}$$

Taking

$$\mathbf{r}\times\mathbf{g} = r\frac{h}{h_0}\mathbf{k}\, e \cos f$$

into account and eliminating T by means of the equation

$$T\mathbf{k}\times\mathbf{r}^* = \mathbf{F} - S\mathbf{r}^* - Z\mathbf{k},$$

we deduce an equation analogous to Equation 26:

$$\frac{d\mathbf{g}}{dt} = \frac{1}{h_0}\left(1 + \frac{r}{p}\right)\mathbf{F} - \frac{1}{h_0}\frac{\mathbf{r}\,\mathbf{r}\cdot\mathbf{F}}{rp} - \frac{2r}{h_0 p}\mathbf{k}\,\mathbf{k}\cdot\mathbf{F}. \tag{45}$$

Integrating the last equation, we obtain the components of $\mathbf{g}$ in the inertial system directly.

Let σ be the distance of the node from the x-axis of the ideal system. Putting (as in Hansen's theory)

$$2N = \sigma_0 + \Omega_0 - \sigma - \Omega - 2\int n_0 \alpha\, dt\ , \tag{46}$$

$$2K = \sigma_0 - \Omega_0 - \sigma + \Omega + 2\int n_0 \eta\, dt\ , \tag{47}$$

where Ω is the longitude of the ascending node of the satellite, we have

$$(\sigma) = \sigma_0 - \int(\alpha - \eta)n_0 dt, \tag{48}$$

$$\left.\begin{aligned} \sigma &= (\sigma) - (K + N), \\ \Omega &= \Omega_0 - \int(\alpha + \eta)n_0 dt + (K - N). \end{aligned}\right\} \tag{49}$$

We obtain from Equation 43:

$$f + \chi - \sigma = f + \chi - \sigma_0 + \int(\alpha - \eta)n_0 dt + N + K.$$

In his previous work (Reference 10) the author made use of the parameters

$$\lambda_1 = \sin\frac{i}{2}\cos N, \qquad \lambda_3 = \cos\frac{i}{2}\sin K,$$

$$\lambda_2 = \sin\frac{i}{2}\sin N, \qquad \lambda_4 = \cos\frac{i}{2}\cos K.$$

With the present notation, the equations for the λ's take the following form:

$$\left.\begin{aligned}
\frac{d\lambda_1}{dt} &= +\, n_0\alpha\lambda_2 + \frac{1}{2}\left[\frac{h}{h_0}\, Zh_0\left(+\lambda_4\ell - \lambda_3 m\right)\right], \\
\frac{d\lambda_2}{dt} &= -\, n_0\alpha\lambda_1 + \frac{1}{2}\left[\frac{h}{h_0}\, Zh_0\left(-\lambda_3\ell - \lambda_4 m\right)\right], \\
\frac{d\lambda_3}{dt} &= +\, n_0\eta\lambda_4 + \frac{1}{2}\left[\frac{h}{h_0}\, Zh_0\left(+\lambda_2\ell + \lambda_1 m\right)\right], \\
\frac{d\lambda_4}{dt} &= -\, n_0\eta\lambda_3 + \frac{1}{2}\left[\frac{h}{h_0}\, Zh_0\left(-\lambda_1\ell + \lambda_2 m\right)\right],
\end{aligned}\right\} \tag{50}$$

where

$$\left.\begin{aligned}
\ell &= r\cos\left[f + \chi - (\sigma)\right], \\
m &= r\sin\left[f + \chi - (\sigma)\right].
\end{aligned}\right\} \tag{51}$$

The development of the equations for the λ's is completely analogous to the development performed by the author in Reference 10 for slightly different definitions of ℓ and m.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors of the inertial frame of reference. Putting

$$A_3(\alpha) = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{e}_3\mathbf{e}_3 + (I - \mathbf{e}_3\mathbf{e}_3)\cos\alpha + I\times\mathbf{e}_3\sin\alpha$$

and

$$A_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & +\cos\alpha \end{bmatrix} = \mathbf{e}_1\mathbf{e}_1 + (I - \mathbf{e}_1\mathbf{e}_1)\cos\alpha + I\times\mathbf{e}_1\sin\alpha,$$

we have

$$\mathbf{r} = A_3(\Omega)\cdot A_1(i)\cdot A_3(\chi-\sigma)\cdot\begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix};$$

or, taking Equations 46-49 into account, we deduce

$$\mathbf{r} = \Gamma \cdot \begin{bmatrix} r\cos(f + \chi - \pi_0 - \int n_0 y\, dt) \\ r\sin(f + \chi - \pi_0 - \int n_0 y\, dt) \\ 0 \end{bmatrix}, \tag{52}$$

where $(\omega) = (\pi_0 - \sigma_0) + \int n_0(y + \alpha - \eta)dt$

and

$$\Gamma = A_3\left[(\Omega)\right] \cdot \Lambda \cdot A_3\left[(\omega)\right], \tag{53}$$

$$\Lambda = I + \frac{2}{1 + s^2}\left[\mathbf{s}\times I + \mathbf{s}\times(\mathbf{s}\times I)\right]$$

$$= A_3(K - N) \cdot A_1(i) \cdot A_3(K + N). \tag{54}$$

Evidently

$$\Gamma = \mathbf{i}\mathbf{e}_1 + \mathbf{j}\mathbf{e}_2 + \mathbf{k}\mathbf{e}_3. \tag{55}$$

In other words the columns of the matrix Γ are the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Introducing the vector

$$\mathbf{s} = \frac{\mathbf{e}_1\lambda_1 - \mathbf{e}_2\lambda_2 + \mathbf{e}_3\lambda_3}{\lambda_4} = \mathbf{e}_1 \frac{\cos N}{\cos K}\tan\frac{i}{2} - \mathbf{e}_2\frac{\sin N}{\cos K}\tan\frac{i}{2} + \mathbf{e}_3 \tan K \tag{56}$$

and putting

$$\mathbf{R} = \mathbf{e}_1 \ell + \mathbf{e}_2 m, \tag{57}$$

we deduce from Equations 50:

$$\frac{d\mathbf{s}}{dt} = n_0 \alpha\, \mathbf{s}\times\mathbf{e}_3 + n_0 \eta(\mathbf{e}_3 + \mathbf{s}\,\mathbf{s}\cdot\mathbf{e}_3) + \frac{1}{2}\frac{h}{h_0} Zh_0(I + \mathbf{s}\times I + \mathbf{s}\,\mathbf{s})\cdot\mathbf{R}. \tag{58}$$

It follows from Equation 54 that

$$\frac{1}{2}(1 + s^2)(\Lambda + I) = I + \mathbf{s}\times I + \mathbf{s}\,\mathbf{s},$$

and Equation 58 takes the form:

$$\frac{d\mathbf{s}}{dt} = n_0 \alpha\, \mathbf{s} \times \mathbf{e}_3 + n_0 \eta (\mathbf{e}_3 + \mathbf{s}\,\mathbf{s} \cdot \mathbf{e}_3) + \frac{1}{4}(1 + s^2)\, \frac{h}{h_0}\, Zh_0 (\Lambda + \mathrm{I}) \cdot \mathbf{R}. \tag{59}$$

We have, from Equations 52 and 53,

$$\mathbf{r} = A_3\big[+(\Omega)\big] \cdot \Lambda \cdot \mathbf{R}$$

and consequently,

$$\mathbf{R} = \Lambda^{T} \cdot A_3\big[-(\Omega)\big] \cdot \mathbf{r}. \tag{60}$$

Substituting Equation 60 into Equation 59 we deduce, finally, that

$$\frac{d\mathbf{s}}{dt} = n_0 \alpha\, \mathbf{s} \times \mathbf{e}_3 + n_0 \eta (\mathbf{e}_3 + \mathbf{s}\,\mathbf{s} \cdot \mathbf{e}_3) + \frac{1}{4}(1 + s^2)\, \frac{h}{h_0}\, Zh_0 \mathbf{r} \cdot A_3\big[(\Omega)\big] \cdot (\mathrm{I} + \Lambda). \tag{61}$$

We have from Equations 39 and 53:

$$\boldsymbol{\rho} = a\Gamma \cdot \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}. \tag{62}$$

This value of $\boldsymbol{\rho}$ must be used in connection with Equation 38. Equations 28 and 29 of the previous section are to be used for the determination of h_0/h and n. It is necessary to emphasize that in this exposition only vectorial and dyadic notations are used, and therefore it is not necessary to distinguish between column vectors and row vectors.

Designating the perturbations of the mean longitude in the orbital plane by ΔL_1, we can write Kepler's equation in the form

$$E - e \sin E = c_0 + n_0 t + \Delta L_1 - (\chi - \pi_0) + \int_0^t \int_0^t \frac{dn}{dt}\, dt^2 \tag{63}$$

or, more conveniently, in the form

$$\lambda - \frac{h_0}{h}(u \sin\lambda - v \cos\lambda) = c_0 + \int_0^t n_0 (1 - y) dt + \Delta L_1 + \int_0^t \int_0^t \frac{dn}{dt}\, dt^2, \tag{64}$$

where u and v are obtained by means of the equation

$$[u, v, 0] = \mathbf{g} \cdot \Gamma. \tag{65}$$

Differentiating Equation 35 we have

$$\frac{d\mathbf{g}}{dt} = -\mathbf{i}\,\frac{dv}{dt} + \mathbf{j}\,\frac{du}{dt} + n_0 y\,\mathbf{k}\times\mathbf{g} + hZ\mathbf{r}\times\mathbf{g}. \tag{66}$$

It follows from the last equation that

$$\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt} = u\,\frac{dv}{dt} - v\,\frac{du}{dt} + n_0 y\, g^2.$$

Substituting this result into

$$\frac{d\chi}{dt} - n_0 y = \frac{1}{g^2}\left(u\,\frac{dv}{dt} - v\,\frac{du}{dt}\right),$$

we obtain

$$\frac{d\chi}{dt} = \frac{1}{g^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt}\ .$$

From the last equation and Equation 30 we deduce a formula of the same form as Equation 32:

$$\frac{d\Delta L_1}{dt} = -\frac{2}{\sqrt{a}}\,\mathbf{r}\cdot\mathbf{F} + \frac{1}{1+\sqrt{1-e^2}}\,\frac{h_0^{\,2}}{h^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt}\ .$$

Setting

$$\frac{d\Delta L}{dt} = n - n_0(1+y) - \frac{2}{\sqrt{a}}\,\mathbf{r}\cdot\mathbf{F} + \frac{1}{1+\sqrt{1-e^2}}\,\frac{h_0^{\,2}}{h^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt}\ , \tag{67}$$

we have

$$\lambda - \frac{h_0}{h}\,(u\,\sin\lambda - v\,\cos\lambda) = c_0 + n_0 t + \Delta L. \tag{68}$$

From the preceding developments we can now give a collection of formulas for the computation of special perturbations of a non-singular set of vectorial elements:

$$\frac{d\mathbf{g}}{dt} = \frac{1}{h_0}\left(1+\frac{r}{p}\right)\mathbf{F} - \frac{1}{h_0}\,\frac{\mathbf{r}\,\mathbf{r}\cdot\mathbf{F}}{rp} - \frac{2r}{h_0 p}\,\mathbf{k}\,\mathbf{k}\cdot\mathbf{F}\,, \tag{69}$$

$$\frac{d\mathbf{s}}{dt} = n_0\alpha\, \mathbf{s}\times\mathbf{e}_3 + n_0\eta(\mathbf{e}_3 + \mathbf{s}\,\mathbf{s}\cdot\mathbf{e}_3) + \frac{1}{4}(1 + s^2)\,\frac{h}{h_0}\,h_0\mathbf{k}\cdot\mathbf{F}\;\mathbf{r}\cdot\Phi, \tag{70}$$

where

$$\Phi = A_3\left[(\Omega)\right](I + \Lambda).$$

We have also

$$\frac{d}{dt}\left(\frac{h_0}{h}\right) = h_0\,\mathbf{F}\cdot\mathbf{k}\times\mathbf{r}, \tag{71}$$

$$\frac{d\Delta L}{dt} = n - n_0(1 + y) - \frac{2}{\sqrt{a}}\,\mathbf{r}\cdot\mathbf{F} + \frac{1}{1 + \sqrt{1 - e^2}}\,\frac{h_0^2}{h^2}\,\mathbf{k}\cdot\mathbf{g}\times\frac{d\mathbf{g}}{dt}\,. \tag{72}$$

The matrices of rotation are computed using the formulas

$$\Lambda = \frac{1 - s^2}{1 + s^2}\,I + \frac{2}{1 + s^2}\,(\mathbf{s}\times I + \mathbf{s}\,\mathbf{s}), \tag{73}$$

$$\Gamma = A_3\left[(\Omega)\right]\cdot\Lambda\cdot A_3\left[(\omega)\right]\,. \tag{74}$$

The remaining elements are computed by means of the equations

$$[u, v, 0] = \mathbf{g}\cdot\Gamma, \qquad a = \frac{a_0(1 - e_0^2)}{\dfrac{h^2}{h_0^2} - g^2}, \qquad e = \frac{h_0}{h}|g|, \tag{75}$$

and the position vector is computed by means of the formulas

$$\boldsymbol{\rho} = a\Gamma\cdot\begin{bmatrix}\cos\lambda\\ \sin\lambda\\ 0\end{bmatrix}, \tag{76}$$

$$\mathbf{r} = \boldsymbol{\rho} - \frac{h_0^2}{h^2}\,\frac{\mathbf{g}\;\mathbf{g}\cdot\boldsymbol{\rho}}{1 + \sqrt{1 - e^2}} + a\frac{h_0}{h}\,\mathbf{k}\times\mathbf{g}\,, \tag{77}$$

$$\lambda - \frac{h_0}{h}\,(u\,\sin\lambda - v\,\cos\lambda) = c_0 + n_0 t + \Delta L\,. \tag{78}$$

APPLICATION TO THE DETERMINATION OF THE RADIATION PRESSURE EFFECT

In the case of the radiation pressure there exists a force function, which can be written in the form

$$\Omega = F\mathbf{u}^0 \cdot \mathbf{r}, \tag{79}$$

where $\mathbf{u}^0$ is the unit vector directed toward the sun. Put

$$\mathbf{u}^0 \cdot \mathbf{i} = A, \quad \mathbf{u}^0 \cdot \mathbf{j} = B, \quad \mathbf{u}^0 \cdot \mathbf{k} = C, \quad \psi = \chi - \pi_0 - \int n_0 y\, dt. \tag{80}$$

Then

$$\mathbf{r} = \mathbf{i} r \cos(f + \psi) + \mathbf{j} r \sin(f + \psi)$$

and

$$\Omega = FrA \cos(f + \psi) + FrB \sin(f + \psi). \tag{81}$$

Equations 41 and 42 can be written in the form (Reference 11)

$$\frac{du}{dt} = + \frac{1}{h_0} \sin(f + \psi) \frac{\partial \Omega}{\partial r} + \frac{1}{h_0 r} \left(1 + \frac{1}{1 - e^2} \frac{r}{a}\right) \cos(f + \psi) \frac{\partial \Omega}{\partial f} + n_0 y v, \tag{82}$$

$$\frac{dv}{dt} = - \frac{1}{h_0} \cos(f + \psi) \frac{\partial \Omega}{\partial r} + \frac{1}{h_0 r} \left(1 + \frac{1}{1 - e^2} \frac{r}{a}\right) \sin(f + \psi) \frac{\partial \Omega}{\partial f} - n_0 y u, \tag{83}$$

where, as before,

$$u = \frac{h}{h_0} e \cos \psi, \qquad v = \frac{h}{h_0} e \sin \psi.$$

Setting

$$w = u + iv, \quad \bar{w} = u - iv, \quad \nu = B + iA, \quad \bar{\nu} = B - iA,$$

where $i = \sqrt{-1}$, and substituting Equation 81 into Equations 82 and 83, we obtain

$$\frac{dw}{dt} = \bar{\nu} \frac{F}{h_0} + \frac{F}{2h_0} \frac{\bar{\nu}}{1 - e^2} \cdot \frac{r}{a} + \frac{F}{2h_0} \frac{\nu}{1 - e^2} \cdot \frac{r}{a} \exp\left[2i(f + \psi)\right] - n_0 i y w. \tag{84}$$

In order to introduce the effect of the earth's shadow we have to find the average disturbing force with respect to the mean anomaly M over the sunlit part of the orbit. Let λ_1 correspond to the point where the satellite leaves the shadow and λ_2 correspond to the point of re-entry. Taking $dM = (r/a)d\lambda$ into account, we deduce from Equations 84

$$\frac{dw}{dt} = \bar{\nu}\frac{F}{h_0}K_1 + \frac{F}{2h_0}\frac{\bar{\nu}K_2 + \nu K_3}{1 - e^2} - in_0 yw, \tag{85}$$

where

$$K_1 = \frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} \frac{r}{a}\, d\lambda, \tag{86}$$

$$K_2 = \frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} \left(\frac{r}{a}\right)^2 d\lambda, \tag{87}$$

$$K_3 = \frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} \left(\frac{r}{a}\right)^2 \exp\left[2i(f + \psi)\right] d\lambda. \tag{88}$$

Set

$$e = \sin\phi, \qquad \xi = \exp i\lambda.$$

Then from the basic formula (Reference 12)

$$\left(\frac{r}{a}\right)^p \exp iqf = \left(\cos^{2p}\frac{\phi}{2}\exp iqE\right)\left(1 - \tan\frac{\phi}{2}\exp iE\right)^{p-q}\left(1 - \tan\frac{\phi}{2}\exp -iE\right)^{p+q}, \tag{89}$$

we deduce

$$\frac{r}{a} = -\frac{1}{2}\frac{h_0}{h}\bar{w}\xi + 1 - \frac{1}{2}\frac{h_0}{h}w\xi^{-1}, \tag{90}$$

$$\left(\frac{r}{a}\right)^2 = \frac{1}{4}\frac{h_0^2}{h^2}\bar{w}^2\xi^2 - \frac{h_0}{h}\bar{w}\xi + \left(1 + \frac{1}{2}e^2\right) - \frac{h_0}{h}w\xi^{-1} + \frac{1}{4}\frac{h_0^2}{h^2}w^2\xi^{-2}, \tag{91}$$

$$\left(\frac{r}{a}\right)^2 \exp 2i(f + \psi) = \frac{1}{4}\left(1 + \sqrt{1 - e^2}\right)^2 \xi^2 - \frac{h_0}{h} w \left(1 + \sqrt{1 - e^2}\right) \xi + \frac{3}{2}\frac{h_0^2}{h^2} w^2$$

$$- \frac{h_0^3}{h^3} w^3 \frac{\xi^{-1}}{1 + \sqrt{1 - e^2}} + \frac{1}{4}\frac{h_0^4}{h^4} w^4 \frac{\xi^{-2}}{\left(1 + \sqrt{1 - e^2}\right)^2} . \tag{92}$$

Substituting Equations 90-92 into Equations 86-88 and taking

$$d\lambda = \frac{d\xi}{i\xi}$$

into account, we deduce

$$K_1 = \frac{1}{2\pi i}\left(- \frac{1}{2}\frac{h_0}{h} \bar{w}\xi + i\lambda + \frac{1}{2}\frac{h_0}{h} w\xi^{-1}\right)_{\lambda_1}^{\lambda_2} , \tag{93}$$

$$K_2 = \frac{1}{2\pi i}\left[+ \frac{1}{8}\frac{h_0^2}{h^2}\bar{w}^2\xi^{+2} - \frac{h_0}{h}\bar{w}\xi + \left(1 + \frac{1}{2}e^2\right) i\lambda + \frac{h_0}{h} w\xi^{-1} - \frac{1}{8}\frac{h_0^2}{h^2} w^2\xi^{-2}\right]_{\lambda_1}^{\lambda_2} , \tag{94}$$

$$K_3 = \frac{1}{2\pi i}\left[+ \frac{1}{8}\left(1 + \sqrt{1 - e^2}\right)^2 \xi^2 - \frac{h_0}{h} w \left(1 + \sqrt{1 - e^2}\right) \xi + \frac{3}{2}\frac{h_0^2}{h^2} w^2 i\lambda \right.$$

$$\left. + \frac{h_0^3}{h^3} w^3 \frac{\xi^{-1}}{1 + \sqrt{1 - e^2}} - \frac{1}{8}\frac{h_0^4}{h^4} w^4 \frac{\xi^{-2}}{\left(1 + \sqrt{1 - e^2}\right)^2}\right]_{\lambda_1}^{\lambda_2} . \tag{95}$$

The presence of a secular effect caused by the earth's shadow (Reference 13) in Equations 93-95 is evident.

Introducing the λ-parameters from the previous section and putting

$$p = \lambda_1 - i\lambda_2, \quad q = \lambda_4 - i\lambda_3,$$

we can write Equations 50 in the form

$$\frac{dp}{dt} = + i n_0 \alpha p + \frac{1}{2}\frac{h}{h_0} Z h_0 \bar{q} r \exp i \left[f + \psi + (\omega)\right] , \tag{96}$$

$$\frac{dq}{dt} = - in_0\eta q - \frac{1}{2}\frac{h}{h_0} Zh_0 \bar{p} r \exp i\left[f + \psi + (\omega)\right], \tag{97}$$

where (ω) is the "mean" argument of the perigee,

$$(\omega) = (\pi_0 - \sigma_0) + \int n_0(\alpha - \eta + y)dt.$$

Setting

$$K_4 = \frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} \left(\frac{r}{a}\right)^2 \exp\left[i(f + \psi)\right] d\lambda,$$

we deduce for the radiation pressure effect:

$$\frac{dp}{dt} = + in_0\alpha p + \frac{F}{2}\frac{h}{h_0} \cdot aCh_0\,\bar{q} \exp i(\omega) \cdot K_4, \tag{98}$$

$$\frac{dq}{dt} = - in_0\eta q - \frac{F}{2}\frac{h}{h_0} \cdot aCh_0\,\bar{p} \exp i(\omega) \cdot K_4, \tag{99}$$

and it follows from Equation 89 that

$$K_4 = \frac{1}{2\pi i}\left[-\frac{1}{8}\left(1 + \sqrt{1 - e^2}\right)\frac{h_0}{h}\bar{w}\xi^2 + \frac{1}{2}\left(1 + \sqrt{1 - e^2}\right)\left(2 - \sqrt{1 - e^2}\right)\xi\right.$$
$$\left. - \frac{3}{2}\frac{h_0}{h} \cdot wi\lambda - \frac{1}{2}\frac{2 + \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}}\frac{h_0^2}{h^2} w^2\xi^{-1} + \frac{1}{8}\frac{h_0^3}{h^3}\frac{w^3\xi^{-2}}{1 + \sqrt{1 - e^2}}\right]_{\lambda_1}^{\lambda_2}.$$

The position of the orbit plane at the time t is given as before by the rotation matrix $\Gamma = A_3\left[(\Omega)\right] \cdot \Lambda \cdot A_3\left[(\omega)\right]$, and the columns of this matrix are the components of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in the inertial system. Thus, the effect of the oblateness perturbations will be concentrated mainly in $A_3\left[(\Omega)\right]$, $A_3\left[(\omega)\right]$. The matrix Λ carries predominantly the effect of the radiation pressure and has the same form as the corresponding matrix introduced in the author's previous work (Reference 10).

From

$$\frac{d}{dt}\left(\frac{h_0}{h}\right) = h_0\frac{\partial\Omega}{\partial f}, \tag{100}$$

we deduce for the radiation pressure effect, taking Equations 81 and 84 into account,

$$\frac{d}{dt}\left(\frac{h_0}{h}\right) = \frac{1}{2} Fh_0 a(\nu K_4 + \bar{\nu}\bar{K}_4). \tag{101}$$

The values of λ_1 and λ_2 are obtained from the equation

$$\mathbf{r} \cdot \mathbf{u}^0 = -\sqrt{r^2 - \rho_1^2}, \tag{102}$$

where ρ_1 is the mean radius of the earth and Equation 38 is substituted for $\mathbf{r}$. The programming of the formulas given in this section can be done by using the complex arithmetic method developed by the IBM Corporation (Reference 14).

CONCLUSION

This paper has presented a new form of the variation of constants, which can be used for numerical integration of the perturbations of planets or satellites even for orbits with very low eccentricities or inclinations. The "small divisor" in the form of e or $\sin i$, which appears in the classical and in some modern methods, is not present in this method.

Recently Dr. Paul Herget published a work which gave a method for solving this problem in a different way. In his method he also used a modified Kepler's equation but he combined the eccentric anomaly with the argument of perigee (Reference 15). In the present work it was found more convenient to combine the eccentric anomaly with the position of the perigee reckoned from the departure point.

In particular, the effect of the radiation pressure on nearly circular orbits (Echo I type) can be treated by using the formulas given in the preceding section of this discussion. In addition, through numerical integration, the effects of oblateness, drag, and radiation pressure can be combined more easily by this method than by a development of the perturbations into trigonometric series.

Extensive and important work has been done by many authors in the field of analytical or semi-analytical theories of artificial satellites. However, many important non-gravitational effects which can be observed in the motion of satellites do not favor the development of perturbations into trigonometric series. In order to include all these effects as well as their cross actions, it would be preferable to resort to a special perturbations method.

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